TECHNICAL REPORT

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November 10, 1989

of work done until Nov. 1989

following papers have been accepted by the publisher.

1. Reduction of 3x3 Polynomial Bundles and New Types of Integrable 3-Wave Interactions, by V.S. Gerdjikov and D.J. Kaup (to appear in the proceedings of the Como Conference of July 5-15, 1988).

2. Nonlinear Modulational Stability of an Electromagnetic Pulse in a Two-Component, Neutral Plasma, by D.J. Kaup and Ronald E. Kates [Accepted by J. Plasma Physics]

   This manuscript describes how to correctly calculate the nonlinear coefficients in the case of an electromagnetic pulse propagating in a two-component plasma. We also demonstrate that other values given in the literature are incorrect. We correct the predictions for such electromagnetic propagation and discuss the astrophysical consequences.

   The above results have been presented as short talks at two meetings, the APS plasma physics meeting in Nov. 1988 and at the Grossman general relativity meeting in Australia in Aug. 1988.

3. The Influemce of an Ambient Magnetic Field on the Nonlinear Modulational Stability of Circularly Polarized Electromagnetic Pulses in a Two-Component, Neutral Plasma, by Ronald Kates and D.J. Kaup [accepted by J. Plasma Physics].

   Here we demonstrate that in the limit of a strong magnetic field, the modulationally stable case in #2 above will become modulationally unstable.

The following work is in various stages of being completed.

4. The Modulational Instability of a Relativistic Electromagnetic Pulse in a Two-Component, Neutral Plasma, by Ronald E. Kates and D.J. Kaup (in preparation).

   This problem has finally simplified. In particular, the longitudinal electric field is found to be a dominate component in this instability. And we can now extend the results on modulational instability from the nonrelativistic domain up into the relativistic domain.

5. A Singular Perturbation Expansion for the Cavity Modes of a Micron-Size Spherical Droplet, by D.J. Kaup [in preparation]

   This paper presents a rapid proceedure for generating analytic solutions for cavity modes of such droplets. Although we have only calculated the eigenfrequencies so far, the potential is now here to <u>analytically</u> evaluate nonlinear interactions in such droplets, and under a varity of conditions.

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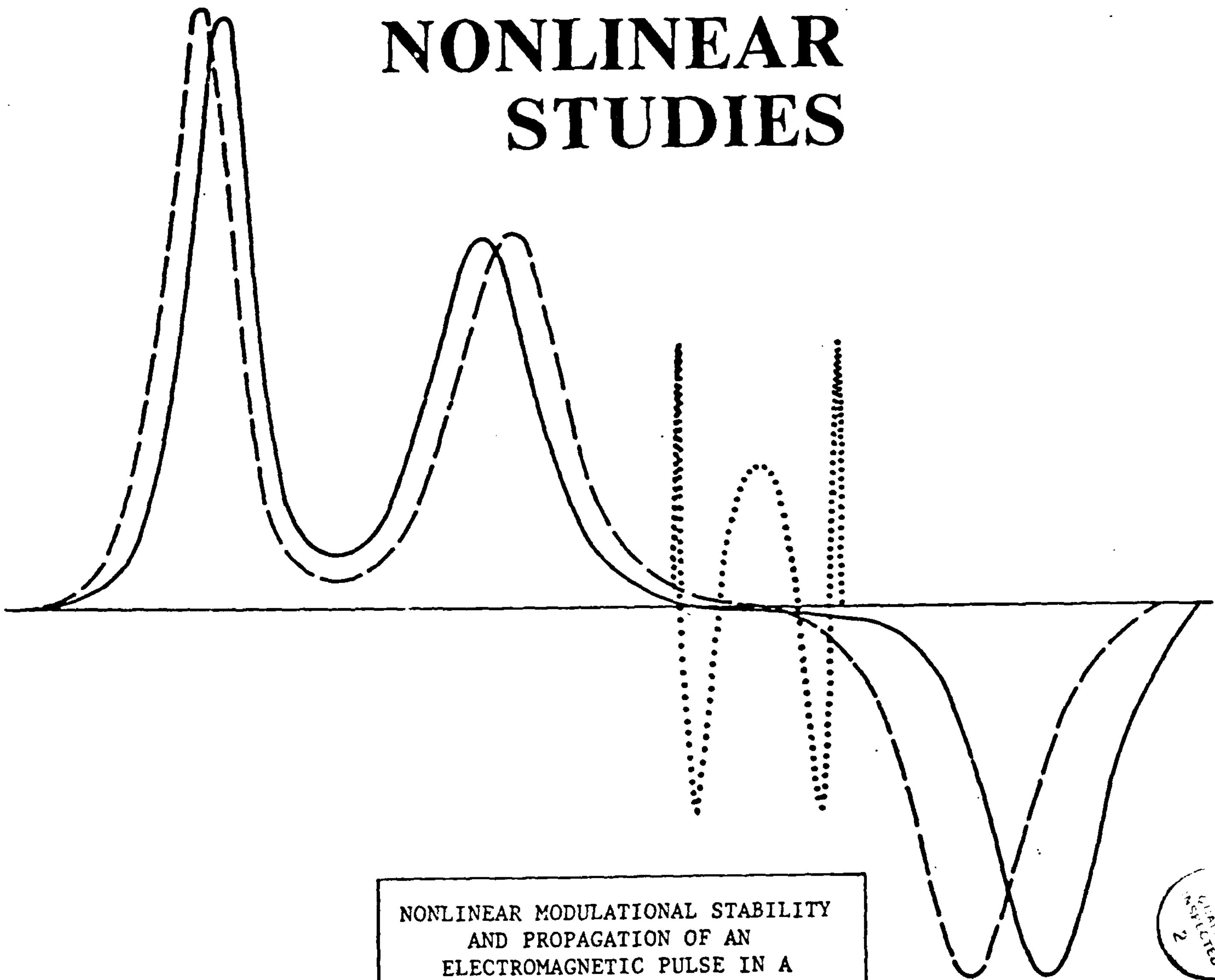

NONLINEAR MODULATIONAL STABILITY AND PROPAGATION OF AN ELECTROMAGNETIC PULSE IN A TWO-COMPONENT, NEUTRAL PLASMA

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Abstract

We study nonlinear effects including possible modulational instability of an intense electromagnetic pulse propagating through a fully-ionized, unmagnetized plasma. (The pulse is assumed to be strong enough to accelerate particles to weakly, but not fully, relativistic velocities.) The envelope is shown to evolve over long time scales in general according to a vector form of the well-known cubic nonlinear Schroedinger (NLS) equation. Three distinct nonlinear effects contribute terms cubic in the amplitude and thus can be of comparable magnitude: ponderomotive forces, relativistic corrections, and harmonic generation. In contrast to previous work, our calculation takes all three effects into account. Integrability and modulational stability of any given system are shown to depend on polarization, frequency, composition, and temperature, and these dependences are given. Finite temperature effects are considered for completeness; they are qualitatively important only near a critical frequency ($< 3/2\, \omega_p$) just above the plasma frequency for which the group velocity is comparable to either (or both) of the sound velocities. In the special case of a cold positron-electron plasma – contrary to the predictions of Chian and Kennel [1] – the model is strictly modulationally stable for both linear and circular polarization; The results have important implications for pulsar micropulse observations and possible technological applications.

The propagation of a relativistically-strong, modulated electromagnetic pulse in a plasma is more complicated than appears at first glance. In the case of uniform propagation (i.e., no modulation), some exact solutions (for circular polarization) are indeed known [2] which are described by a nonlinear dispersion relation. Chian and Kennel [1] pointed out the possible application to pulsar micropulses and obtained a scalar NLS equation by applying the method of Karpman and Krushkal [6] (which is based on the Whitham averaging method [7]) to this dispersion relation. The problem was again considered by Mofiz and collaborators [3-5] Unfortunately, the resulting NLS coefficients in these papers are incorrect, and the force equations in [1] are violated at leading order. (The necessity of paying special attention to the longitudinal force equations was noticed in [3].) The problem evidently contains subtleties and requires the careful application of singular perturbation methods.

The scalar NLS equation is given in standard dimensionless form by

$$i\,\partial_t q = -\partial_x^2 q \pm 2q^*\, q^2 \quad . \tag{1}$$

The upper sign corresponds to modulational instability. As we shall see, the propagation of the envelope of an electromagnetic pulse in an unmagnetized, neutral, two-component plasma in the weakly nonlinear case is governed in general, not by (1), but by a vector form of the NLS equation. By weakly nonlinear we mean that the transverse electric field accelerates one or (possibly) both components to weakly relativistic velocities within one cycle, that is,

$$\frac{e}{m_e c\,\omega}|\vec{E}| \lesssim 1 \quad , \tag{2}$$

In the general case, two NLS amplitudes will be present, corresponding to the two polarization states. However, these polarization states will be coupled by the nonlinearity. Consider a system of the form

$$i\partial_\tau q_1 + D\partial_\xi^2 q_1 + q_1(C_1 q_1 q_1^* + C_2 q_2 q_2^*) = 0 \tag{3a}$$

$$i\partial_\tau q_2 + D\partial_\xi^2 q_2 + q_2(C_1 q_2 q_2^* + C_2 q_1 q_1^*) = 0 \tag{3b}$$

In the special case where C_1 and C_2 are equal (or can be transformed into this form), the equations are known to be integrable [8]. However, in general, the vector NLS equation will be nonintegrable.

The purpose of this paper is to present an accurate evaluation of the leading nonlinear coefficients, because they are crucial in determining the qualitative behavior of the envelope, in particular whether or not modulational stability and soliton type solutions will occur. The reasons for performing this calculation are several fold. For example, with the advent of current technological capabilities, one could be interested in how nonlinear effects will couple different polarizations, particularly if one intends to have each polarization mode carry different bits of information. Also, intense signals in a rarified plasma could have critical information scrambled by the nonlinear effects. It may even be possible to test our theory in astrophysical systems, in particular by observing pulsar micropulses. This point will be discussed by us in a separate paper [9].

In any singular perturbation method, one seeks uniformly valid asymptotic expansions [10] with respect to a small parameter, which we will call ϵ (in this problem ϵ is related to the typical amplitude of the vector potential). Here, the domain of uniform validity is of order $1/\epsilon$ cycles in space and $1/\epsilon^2$ cycles in time. The singular perturbation method which we will apply, known as the method of multiple time scales, has been described in several textbooks (see for example Van Dyke [11] or Nayfeh [12]) and has been applied to a wide variety of problems. Since we are interested in the evolution of a pulse with arbitrary modulation, it will not be possible to assume dependence on only a single independent variable (although it is possible to verify consistency of our results for particular cases with the work of Kozlov, et. al. [13], in which fully nonlinear solutions depending on one independent variable were found). Instead, we allow the functions in this problem to depend, not only on a "fast" (phase) variable, but also on "slow" time and space variables. These will be chosen to make the effects of nonlinearity enter the calculation at the same order as the effects of linear spreading of the wave packet.

Consider an envelope of length L of waves with wavelength near λ. If $\epsilon \equiv \frac{e}{mc\omega}|E| \sim \frac{\lambda}{L}$, then nonlinear effects will typically act on a timescale $T_0 = \lambda/(\epsilon^2 c)$ comparable to the timescale for the spreading of packet due to linear dispersion. By routine dimensional analysis it is possible to see that ponderomotive forces, relativistic corrections, and harmonic generation might have comparable effects on this timescale. Our calculation shows that, in general, all three effects indeed play a role.

The ponderomotive force depends quadratically on the amplitude and leads to a slowly varying longitudinal field, corresponding physically to radiation pressure, which leads to slow longitudinal motions and modifies the background density. At cubic order, the modification of the background couples back to the fundamental. This contribution dominates at the longest wavelengths ($c^2k^2 \ll \omega_p^2$). (In fact, for finite temperature, there is a singularity at a critical frequency just above the plasma frequency.) At shorter wavelengths ($c^2k^2 \gtrsim \omega_p^2$), the other two contributions dominate in an electron-positive ion plasma, while all three effects are important in an electron-positron plasma. Independently of the composition, temperature, or frequency, the effects of harmonic generation are identically zero for circular polarization – but in general only for circular polarization.

Chian and Kennel [1] first proposed that modulational instabilities in an electron-positron plasma would reshape pulsar micropulse signals. Unfortunately, they omitted two "parametric" sources of nonlinearity (harmonic generation and ponderomotive effects) and also obtained an incorrect value for the linear dispersion coefficient of the NLS. As we shall see, the special case of an electron-positron plasma exhibits several peculiarities, such as accidental (near) cancellations in certain limits. One can obtain the correct nonlinear coefficient only by considering all three sources of nonlinearity right from the beginning. When all terms are included, we find that, at least in the nonrelativistic limit (their Eq. (10)), their claims concerning modulational instability are reversed – both for linear and for circular polarization. A cold positron-electron plasma is modulationally stable for both circular and linear polarization. However, if ions are present, modulational instability is possible, as we shall see. Implications for pulsar signals will be briefly discussed below.

Our model comprises a neutral, fully-ionized plasma, consisting of singularly-ionized atoms (of charge $+e$ and mass M) and electrons (of charge $-e$ and mass m); the plasma is modeled as a fluid with constant but distinct electron and ion temperatures (possibly zero). The fluid is characterized by the electron density n, the electron velocity $\vec{v}$, the ion density N and the ion velocity $\vec{V}$. (Although we will refer to the positive component as "the ions," our model of course also describes a positron-electron plasma if $m = M$.) The plasma is further characterized by a vector potential $\vec{A}$ and an electrostatic potential ϕ. The functions n, $\vec{v}$ N, $\vec{V}$, $\vec{A}$, and ϕ are assumed to depend on t and z, but not on x and y. It is convenient to write the continuity and Euler equations for the electrons in the form

$$\partial_t n + \vec{\nabla} \cdot (n\vec{v}) = 0 \tag{4a}$$

$$d(m\gamma\vec{v} - \frac{e}{c}\vec{A}) - e\vec{\nabla}\phi + \frac{e}{c}\Sigma_{\ell=1}^{3} v_\ell \vec{\nabla} A_\ell - mc_s^2 \vec{\nabla} \ln n = 0 \tag{4b}$$

where c_s is the electron speed of sound, m is the electron mass, and

$$d = \partial_t + \vec{v} \cdot \vec{\nabla} \tag{4c}$$

$$\gamma = (1 - v^2/c^2)^{-1/2} \tag{4d}$$

For the ions, the respective equations are

$$\partial_t N + \vec{\nabla} \cdot (N\vec{V}) = 0 \tag{5a}$$

$$D(M\Gamma\vec{V} + \frac{e}{c}\vec{A}) + e\vec{\nabla}\phi - \frac{e}{c}\Sigma_{\ell=1}^{3} V_\ell \vec{\nabla} A_\ell - MC_s^2 \cdot \vec{\nabla} \ln N = 0 \tag{5b}$$

where C_s is the ion speed of sound, M is the ion mass, and

$$D = \partial_t + \vec{V} \cdot \vec{\nabla} \tag{5c}$$

$$\Gamma = (1 - V^2/c^2)^{-1/2} \tag{5d}$$

Here, the electron speed of sound c_s is assumed constant and is related to the temperature T_e by $c_s^2 = \gamma_e T_e/m$, and similarly for the ions.

Of course Maxwell's equations must hold:

$$\vec{\nabla} \times \vec{B} = \frac{4\pi e}{c}(n\vec{v} - N\vec{V}) + \frac{1}{c}\partial_t \vec{E} \tag{6a}$$

$$\vec{\nabla} \cdot \vec{E} = 4\pi e(N - n) \tag{6b}$$

with

$$\vec{B} = \vec{\nabla} \times \vec{A} \tag{6c}$$

$$\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\partial_t \vec{A} \tag{6d}$$

and where we impose the radiation gauge

$$\vec{\nabla} \cdot \vec{A} = 0 \quad . \tag{6e}$$

This is a nonlinear system of equations is to be solved for the evolution of some initial data given on the interval $-\infty < z < \infty$ for the functions $(n,\ \vec{v},\ N,\ \vec{V},\ \vec{A},\ \phi)$, satisfying the usual initial-value constraints. (The constraints will be satisfied automatically at later times). We are interested in a particular class of solutions, namely those which correspond to the physical idea of a slowly modulated, weak, nearly sinusoidal disturbance about a uniform quiescent medium (which is an obvious exact solution of Eqs (4-6)). We assume that well-posed initial data can be given which evolve in this way.

To make these assumptions precise, consider first the electron density and the vector potential, which are assumed to have asymptotic expansions of the form

$$n = n_0 + \epsilon n_1 + \epsilon^2 n_2 + \epsilon^3 n_3 + \dots \tag{7a}$$

$$\vec{A} = \qquad \epsilon \vec{A_1} + \epsilon^2 \vec{A_2} + \epsilon^3 \vec{A_3} + \dots \tag{7b}$$

with respect to the positive dimensionless parameter $\epsilon \ll 1$, where the first term n_0 in (7a) is a constant. Since the velocities will also scale with ϵ, it is evident that for sufficiently small ϵ the motion is only weakly relativistic and thus we will be able to expand the relativistic factors γ and Γ about unity.

Similarly, the other variables are expanded as follows:

$$N = n_0 + \epsilon N_1 + \epsilon^2 N_2 + \epsilon^3 N_3 + \dots \tag{7c}$$

$$\vec{v} = \qquad \epsilon \vec{v_1} + \epsilon^2 \vec{v_2} + \epsilon^3 \vec{v_3} + \dots \tag{7d}$$

$$\vec{V} = \qquad \epsilon \vec{V_1} + \epsilon^2 \vec{V_2} + \epsilon^3 \vec{V_3} + \dots \tag{7e}$$

$$\phi = \qquad \epsilon \phi_1 + \epsilon^2 \phi_2 + \epsilon^3 \phi_3 + \dots \tag{7f}$$

Note that $n_0 = \text{const}$, $N_0 = n_0$, and all other zeroth-order quantities vanish. (Except where otherwise stated, numerical subscripts indicate the formal order in ϵ.)

Now, it is routine to work out the <u>linearized</u> solutions of (4)-(6) corresponding to a sum over Fourier components of transverse electromagnetic EM waves. (By "linearized" solutions, we mean solutions obtained by simply truncating (4)-(6) at order ϵ.) A typical Fourier component of the vector potential perturbation behaves in this linearized theory like $\vec{a}(\omega)e^{i(kz-\omega t)}$, where k and ω are related by a dispersion relation to be given below. Here we are interested in the <u>nonlinear</u> theory in which different Fourier components are coupled. In order to study the evolution of a slowly modulated beam, we assume that the first-order perturbation of the vector potential takes the form

$$\vec{A_1} = \vec{a} e^{i(kz-\omega t)} + c.c. \tag{8}$$

where "c.c." means the complex conjugate, k is the wave number, ω is the frequency. The functions a_x and a_y (where the subscripts here refer to the component) are not strictly constant, but depend on the slow variables

$$Z \equiv \epsilon z, \quad T \equiv \epsilon t, \quad \tau \equiv \epsilon^2 t \tag{9}$$

With the introduction of slow coordinates, we employ the following standard procedure: We collect terms in all equations according to their formal order of magnitude. ("Order" always means with respect to ϵ.) We then solve order by order. Every derivative of a slowly varying function is down by one or two orders. For this reason, the transverse equations at leading order are the same as those of the truncated linearized theory. (Not so for the longitudinal equations, as we shall shortly see.) Slow derivatives of the lower-order terms will appear at higher orders. According to standard singular-perturbation procedure [12], we seek solutions of the higher-order equations which satisfy the requirement of uniform validity, not only over a cycle, but also over the slow scales. (In general, not all of the higher-order equations need to be solved explicitly, but instead provide integrability conditions which affect the lower orders.) In this way, the evolution of the slowly varying functions will be determined.

With this strategy in mind, we now solve the first-order equations. The longitudinal components are identically zero or can be chosen to vanish without loss of generality. The transverse components can be treated as a coupled linear homogeneous system of the form

$$L(\psi_1) = 0 \quad . \tag{10}$$

At this order, we choose the EM mode to dominate and take the solution to be of the form

$$\vec{v}_1 = \frac{e}{mc}\vec{A}_1 \tag{11a}$$

$$\vec{V}_1 = \frac{-e}{Mc}\vec{A}_1 \tag{11b}$$

$$\phi_1 = 0 = n_1 = N_1 = A_{1z} = v_{1z} = V_{1z} \tag{11c}$$

Note that the velocities and the vector potential are purely transverse in first order. The first-order part of (6b) guarantees quasineutrality at first order. The z-component of (6a) is then satisfied at first order automatically.

The dispersion relation is

$$\omega^2 = \omega_p^2 + c^2k^2 \tag{12}$$

where

$$\omega_p^2 = 4\pi e^2 n_0 \left(\frac{1}{m} + \frac{1}{M}\right) \tag{13}$$

The slow dependence of $\vec{a}$ is of course as yet undetermined

As we have seen, the longitudinal components of the velocities vanish in first order. However, for a system with slowly modulated waves, the third-order part of the longitudinal component of the force equations (4b) and (5b) leads to a nonvanishing, slowly varying longitudinal component of the velocities at second order. This happens even in a positron-electron plasma, despite the fact that the third-order longitudinal electric field (second-order electrostatic potential) vanishes. Note that in Ref. [1], the longitudinal component of the force equations (their Eq. (5)) is violated in leading order.

At second order there are three possible types of terms with different phase dependence, which we will discuss in turn. Decomposing the second-order electron density n_2 as

$$n_2 = n_2^{(0)} + \left[n_2^{(1)} e^{i(kz-\omega t)} + \text{c.c.}\right] + \left[n_2^{(2)} e^{2i(kz-\omega t)} + \text{c.c.}\right] \quad , \tag{14}$$

we note that the first term $n_2(0)$ corresponds to a second-order correction to the background density. A similar decomposition will be made for the other variables. From here on superscripts (0), (1), (2), etc. refer to coefficients of terms with corresponding phase dependence. Terms such as $n_2^{(0)}$ will be refered to as "DC" terms. Terms such as $n_2^{(1)}$ correspond to a correction to the fundamental. Terms such as $n_2^{(2)}$ correspond to the harmonic, which in general arises due to nonlinearity, although in specific cases it may vanish identically.

At all orders $p > 1$, the fundamental mode is described by an inhomogeneous system of the general form

$$L(\psi_p) = N_p(\psi_1, \ldots \psi_{p-1}) \quad , \tag{15}$$

where L is the linear operator of Eq. (10), and where N_p is a (possibly) nonlinear functional of the lower-order terms. If one defines the scalar product

$$\langle f, g\rangle \equiv \int_0^{2\pi} d\theta\, f^* g \tag{16}$$

of two functions f and g over a cycle, then N_p must be orthogonal (with respect to (16)) to the general homogeneous solution ψ_H of (10). Otherwise, secular terms, e.g., terms of the form $\theta e^{i\theta}$ ($\theta \equiv kx - \omega t$), would arise. Secular terms would obviously violate the uniformity condition stated above, because the pth order would eventually become comparable in size to the terms of order $p-1$. At second order, this requirement implies simply

$$(\partial_T + v_g \partial_Z)\, \vec{a} = 0 \tag{17}$$

where

$$v_g = \frac{d\omega}{dk} = \frac{c^2 k}{\omega} \tag{18}$$

Note that Eq. (17) is the same as the group velocity condition for a wave packet in the truncated linear theory. (Nonlinearites will play a role in the fundamental at third order.) Now that the secular terms in the fundamental mode have been removed, then the solution for the second-order fundamental can be taken to vanish without loss of generality (a nonvanishing contribution could always be absorbed into first order.) Thus

$$\begin{aligned} n_2^{(1)} &= N_2^{(1)} = \phi_2^{(1)} = 0 \\ \vec{v_2}^{(1)} &= \vec{v_2}^{(1)} = \vec{A_2}^{(1)} = 0 \quad . \end{aligned} \tag{19}$$

Now for the second order DC terms. These are obtained by solving z-components of the Euler equations (4b) and (5b) in third order together with the continuity equations

(4a) and (5a) at second order.

$$n_2^{(0)} = N_2^{(0)} = \frac{e^2 n_0}{Mm\, c^2 v_g^2} \vec{a}^* \cdot \vec{a} \tag{20a}$$

$$\vec{v}_2^{(0)} = \vec{v}_2^{(0)} = \hat{z}\ v_g n_2^{(0)}/n_0 \tag{20b}$$

$$\vec{A}_2^{(0)} = 0 \tag{20c}$$

$$\phi_2^{(0)} = \frac{e}{c^2}\left(\frac{1}{m} - \frac{1}{M}\right)\vec{a}^* \cdot \vec{a} \tag{20d}$$

Note that in all cases the plasma acquires a longitudinal component of the velocity in the z-direction, even though $\phi_2^{(0)}$ vanishes for an electron-positron plasma. Except for the electron-positron case, the DC electric field will induce charge separation over the slow space scales at $O(\epsilon^4)$.

The nonlinear terms in the Euler equations (4b) and (5b) also induce harmonic terms in the solution:

$$n_2^{(2)} = \frac{n_0 e^2 k^2 (\vec{a} \cdot \vec{a})}{2m\, \omega^2 c^2 (4\omega^2 - \omega_p^2)} \left(\frac{4\omega^2}{m} - \frac{\omega_p^2}{M}\right) \tag{21a}$$

$$N_2^{(2)} = \frac{n_0 e^2 k^2 (\vec{a} \cdot \vec{a})}{2M\, \omega^2 c^2 (4\omega^2 - \omega_p^2)} \left(\frac{4\omega^2}{M} - \frac{\omega_p^2}{m}\right) \tag{21b}$$

$$\vec{v}_2^{(2)} = \frac{\omega n_2^{(2)}}{k n_0}\hat{z} \tag{21c}$$

$$\vec{V}_2^{(2)} = \frac{\omega N_2^{(2)}}{k n_0}\hat{z} \tag{21d}$$

$$\phi_2^{(2)} = \frac{-e\omega_p^2 \left(\frac{1}{m} - \frac{1}{M}\right)(\vec{a} \cdot \vec{a})}{2c^2(4\omega^2 - \omega_p^2)} \tag{21e}$$

$$\vec{A}_2^{(2)} = 0 \tag{21f}$$

Evidently from Eqs. (21), circular polarization (vanishing $\vec{a} \cdot \vec{a}$) is a special case (for any mass ratio), because harmonic generation is absent. In this case, charge quasi-neutrality is maintained at second order. However, in general the second-order density perturbations of electrons and ions are not equal: charge separation over the fast scales is already present at second order!

The electron-positron case is again special because the harmonic, second order potential perturbation is zero. Otherwise, a second-order electric field varying as the harmonic will be present in general (except for circular polarization).

We now proceed to third order. It is now possible to derive the evolution of the slow functions ($\vec{a}$, etc.) as a consequence of the requirement that the third-order equations contain no secular terms, as explained above. Since DC and harmonic terms are of course orthogonal to the operator L of Eq. (15), it sufficies to consider the fundamental. As at second order, one can absorb the perturbations in the fundamental mode of the third-order electron density $n_3^{(1)}$, ion density $N_3^{(1)}$, and potentials $\phi_3^{(1)}$, $\vec{A}_3^{(1)}$ into the first-order terms without loss of generality. Thus,

$$\phi_3^{(1)} = N_3^{(1)} = n_3^{(1)} = 0, \quad \vec{A}_3^{(1)} = 0 \quad . \tag{22a}$$

However, there remains a third-order perturbation of the velocities of the form which comes from the relativistic corrections

$$\vec{v}_3^{(1)} = \frac{-e^3}{2m^3c^5}\left[2\vec{a}(\vec{a}^* \cdot \vec{a}) + \vec{a}^*(\vec{a} \cdot \vec{a})\right] \tag{22b}$$

$$\vec{V}_3^{(1)} = \frac{+e^3}{2M^3c^5}\left[2\vec{a}(\vec{a}^* \cdot \vec{a}) + \vec{a}^*(\vec{a} \cdot \vec{a})\right] \tag{22c}$$

The secular condition then becomes

$$\begin{aligned} &2i\omega\partial_\tau\vec{a} + (c^2\partial_Z^2 - \partial_T^2)\vec{a} - \omega_p^2\vec{a}(n_2^{(0)}/n_0) \\ &- \omega_p^2\vec{a}^*\left[\frac{n_2^{(2)}}{mn_0} + \frac{N_2^{(2)}}{Mn_0}\right]\frac{1}{\frac{1}{m} + \frac{1}{M}} \\ &+ \frac{e^2\omega_p^2}{2c^4}\,\frac{\frac{1}{m^3} + \frac{1}{M^3}}{\frac{1}{m} + \frac{1}{M}}\left[2\vec{a}(\vec{a}^* \cdot \vec{a}) + \vec{a}^*(\vec{a} \cdot \vec{a})\right] = 0 \end{aligned} \tag{23}$$

The origin of the various terms in (23) is evident. The first two terms would be present even in the linearized system obtained by simple truncation of Eqs. (4-6); they represent the effects of linear dispersion. The term containing $n_2^{(0)}$ arises because the ponderomotive force induces perturbations of the "background" density on the slow scales, as discussed above. The terms containing $n_2^{(2)}$ and $N_2^{(2)}$ arise from harmonic generation. Finally, the last term comes from relativistic momentum corrections contained in the factors of γ and Γ of Eqs. (4) and (5).

Taking into account Eqs. (17), (18), (20), and (21), we can rewrite (23) in the form

$$\begin{aligned} &2i\omega\partial_\tau\vec{a} + \frac{\omega_p^2c^2}{\omega^2}\partial_Z^2\vec{a} + \vec{a}^*(\vec{a} \cdot \vec{a})(C_H + C_R) \\ &+ (C_P + 2C_R)\vec{a}(\vec{a}^* \cdot \vec{a}) = 0 \end{aligned} \tag{24}$$

In the above, the subscripts H (harmonic), P (ponderomotive), and R (relativistic) indicate the origin of the corresponding nonlinear coefficient. These nonlinear coefficients are given for zero temperature by

$$C_H = \frac{-e^2k^2\omega_p^2}{2\omega^2c^2(4\omega^2 - \omega_p^2)}\left[4\omega^2\frac{\frac{1}{m^3} + \frac{1}{M^3}}{\frac{1}{m} + \frac{1}{M}} - \frac{\omega_p^2}{mM}\right] \tag{25a}$$

$$C_P = -\frac{e^2\omega_p^2}{Mm\,c^2v_g^2} \tag{25b}$$

$$C_R = \frac{e^2\omega_p^2}{2c^4}\,\frac{\frac{1}{m^3}+\frac{1}{M^3}}{\frac{1}{m}+\frac{1}{M}} \tag{25c}$$

If the temperature is finite, the relativistic coefficient remains unchanged, whereas Eqs. (25a) and (25b) generalize to

$$C_H = -\frac{e^2k^2\omega_p^2}{2c^2}\left(\frac{mM}{M+m}\right)\left\{\frac{1}{m^3(\omega^2-k^2c_s^2)}+\frac{1}{M^3(\omega^2-k^2C_s^2)}\right.$$
$$\left.+\frac{\omega_p^2}{4D_2}\left(\frac{mM}{M+m}\right)\left[\frac{1}{m^2(\omega^2-c_s^2k^2)}-\frac{1}{M^2(\omega^2-C_s^2k^2)}\right]^2\right\} \tag{25d}$$

with

$$D_2 \equiv 1-\frac{1}{4}\omega_p^2\frac{mM}{M+m}\left[\frac{1}{m(\omega^2-c_s^2h^2)}+\frac{1}{M(\omega^2-C_s^2k^2)}\right]$$

and

$$C_P = -\frac{e^2}{c^2}\omega^2\left(\frac{M+m}{Mm}\right)\frac{1}{M(v_g^2-C_s^2)+m(v_g^2-c_s^2)} \tag{25e}$$

(Note that the factor D_2 varies between 3/4 and 1.) In the high-wavenumber limit, all three coefficients approach constants, whether or not the temperature is finite. In the case of zero temperature, as k approaches zero, the ponderomotive term dominates, because the coefficient C_P diverges like $k^{(-2)}$, whereas the relativistic coefficient C_R approaches a constant and the harmonic coefficient C_H vanishes like k^2. If the temperature is finite, the ponderomotive coefficient C_P is singular just above the plasma frequency. (Since the speed of sound can never exceed $1/\sqrt{3}c$, this critical frequency lies below $3/2\ \omega_p$).

In the cases of linear and circular polarization, the vector NLS reduces to a scalar NLS. For circular polarization, we obtain

$$2i\omega\partial_\tau a+\frac{\omega_p^2c^2}{\omega^2}\partial_Z^2 a+2(C_P+2C_R)a^*a^2=0 \tag{26}$$

where we have set $\vec{a}=a\cdot(\hat{x}+i\hat{y})$. For linear polarization, we obtain

$$2i\omega\partial_\tau a+\frac{\omega_p^2c^2}{\omega^2}\partial_Z^2 a+(C_H+3C_R+C_p)a^*a^2=0 \tag{27}$$

where now we have taken $\vec{a}=a\cdot\hat{x}$. Thus, modulational instability for these two cases occurs if the combinations (C_P+2C_R), $(C_H+3C_R+C_P)$, respectively are positive [8].

We now consider some special cases:

ELECTRON-POSITIVE ION PLASMA: If the plasma is composed of positive ions and electrons, then $m/M \ll 1$ and we find (for the zero-temperature case)

$$C_H+C_R \doteq \frac{3e^2\omega_p^4}{2m^2c^4(4\omega^2-\omega_p^2)}>0 \tag{28a}$$

$$C_P+2C_R \doteq \frac{e^2\omega_p^2}{c^4m^2}\left[1-\frac{m\omega^2}{Mc^2k^2}\right] \tag{28b}$$

Except for frequencies very near the plasma frequency, both polarizations are evidently modulationally unstable. The corresponding results for finite temperature are:

$$C_H + C_R \doteq \frac{3e^2\omega_p^4(1 - 4/3c_s^2h^2/\omega_p^2)}{2m^2c^4(4\omega^2 - \omega_p^2 - 4c_s^2h^2)} \tag{28c}$$

$$C_p + 2C_R \doteq \frac{e^2\omega_p^2}{c^4m^2}\left(1 - \frac{m}{M}\frac{\omega^2}{\omega^2(1 - C_s^2/c^2) - \omega_i^2}\right) \tag{28d}$$

As in the zero-temperature case, both polarizations are modulationally unstable - except for a small frequency range just above the plasma frequency. At finite temperature, the singularity in the ponderomotive coefficient near the plasma frequency leads to corresponding singularities for both linear and circular polarization; sufficiently near the plasma frequency, the system is modulationally stable. On the other hand, finite temperature makes virtually no difference in the behavior of either polarization at moderate to high frequencies - even though the combination $C_H + C_R$ can become negative. In all cases, the contribution of the ponderomotive coefficient is down by $O(m/M)$ compared to the others, except near the plasma frequency. The positive (destabilizing) contribution of the relativistic coefficient outweighs the negative (stabilizing) contribution of the harmonic coefficient.

ELECTRON-POSITRON PLASMA: Next, let us consider an electron-positron plasma, as in Chian and Kennel [1]. Setting $M = m$ in Eqs. (25), we find in the zero-temperature case

$$C_H + C_R = \frac{e^2\omega_p^4}{2m^2c^4\omega^2} > 0 \tag{29a}$$

$$C_P + 2C_R = \frac{-e^2\omega_p^4}{m^2c^6k^2} < 0 \tag{29b}$$

$$C_H + 3C_R + C_p = -\frac{e^2\omega_p^2}{2m^2c^6k^2\omega^2}(\omega^2 + \omega_p^2) \tag{29c}$$

Since $C_H + 3C_R + C_P < 0$ and $C_P + 2C_R < 0$, both polarizations are evidently modulationally stable, contrary to the claim of Chian and Kennel [1]. As noted above, their solution violates the z-component of the force equation at leading order and thus ignores ponderomotive effects, which (as we saw) are stabilizing for both polarizations. For linear polarization, the additional stabilizing effect of mode-mode interactions (harmonic effect) was apparently also neglected.

If the electrons and positrons have finite (but equal) temperatures, we find

$$C_p + 2C_R = \frac{e^2\omega_p^2}{m^2c^4}\left(1 - \frac{c^2}{v_g^2 - c_\pm^2}\right) \tag{29e}$$

$$C_P + 3C_R + C_h = \frac{e^2\omega_p^2}{m^2c^4}\left[\frac{1}{2}\left(\frac{\omega_p^2 - c_\pm^2 k^2}{\omega^2 - c_\pm^2 k^2}\right) - \left(\frac{\omega_p^2 + c_\pm^2/c^2\omega^2}{c^2k^2 - c_\pm^2/c^2\omega^2}\right)\right]$$

$$= \frac{e^2\omega_p^2}{m^2c^4}\left[\frac{1}{2}\left(\frac{(1 + c_\pm^2/c^2)\omega_p^2 - c_\pm^2/c^2\omega^2}{(1 - c_\pm^2/c^2)\omega^2 + c_\pm^2/c^2\omega_p^2}\right)\right. \tag{29f}$$

$$\left. - \left(\frac{\omega_p^2 + c_\pm^2/c^2\omega^2}{\omega^2\left(1 - \frac{c_\pm^2}{c^2}\right) - \omega_p^2}\right)\right]$$

where $c_\pm$ is the speed of sound of both components. Due to the singularity in the ponderomotive coefficient, modulational instability in a electron-positron plasma is possible, but only in a narrow range just above the plasma frequency.

We now briefly discuss astrophysical implications: Recent observational data [14-16] strongly support the hypothesis that pulsar micropulses are a "temporal" phenomenon and can be interpreted within the amplitude-modulated noise model [17]. As discussed in [2-5], nonlinear modulational instability would provide a natural mechanism for amplitude modulations. Now, according to pulsar models [18-19], the pulsar magnetosphere is composed of secondary electrons and positrons. However, this paper shows that, in the absense of an ambient magnetic field, a cold electron-positron plasma does not exhibit modulational instability for linear or circular polarization and that finite temperature affects this result only just above the plasma frequency. On the other hand, we have found that a cold electron positive-ion plasma is modulationally unstable for either polarization. So if pulsar micropulses really are caused by modulational instability, then at least one new feature must enter the problem.

We would like to mention three possible candidates for this new feature and of interest for pulsar micropulses: First, from our above result on the modulational instability of an electron-positive ion plasma, one would expect that the inclusion of some positive ions could drive the electron-positron plasma modulationally unstable. In a three-component neutral plasma consisting of electrons, positrons, and positive ions, we expect that a critical ratio (positive-ion density)/(positron density) will exist - depending perhaps on parameters such as the frequency - since the limiting cases (electron-positron, electron-ion) give opposing results.

The second physical effect would be the presence of ambient magnetic fields, which could strongly affect the modulational instability of any plasma, in particular an electron-positron plasma. Careful calculations are required in order to determine how such a magnetic field would affect the nonlinear coefficients, keeping in mind that new parametric interactions can occur due to the presence of the ambient magnetic field. (Such a calculation was proposed in [5], but unfortunately the treatment contains errors, as can be confirmed for example by comparison of Eq. (7) of [5] with Eq (44) of [20].)

Third, in pulsar magnetospheres, the actual estimated value of the strength parameter of Eq. (2) may be of order unity or larger. Under these circumstances, the present theory cannot be applied directly, since higher-order corrections to the nonlinear coefficients found here could be important, and at the moment, no a priori method for estimating their size is available. A fully relativistic scheme allowing arbitrary (slow) modulations of a zeroth-order solution with large (relativistic) amplitudes would be optimal. However, such a scheme will contain all the subtleties of the present case together with possibly new ones. (For example, slowly varying "DC" ponderomotive terms in the longitudinal force equations will enter two orders in ϵ sooner than in the present calculation.) Now, as we saw above, the Karpman-Krushkal [6] application of Whitham theory [7] as applied in [1] and [3-5] unfortunately did not give satisfactory results in the present problem. But we do note that if the Whitham method could be extended to include parametric interactions, then it might well be possible to obtain estimates of nonlinear coefficients even in the fully relativistic case.

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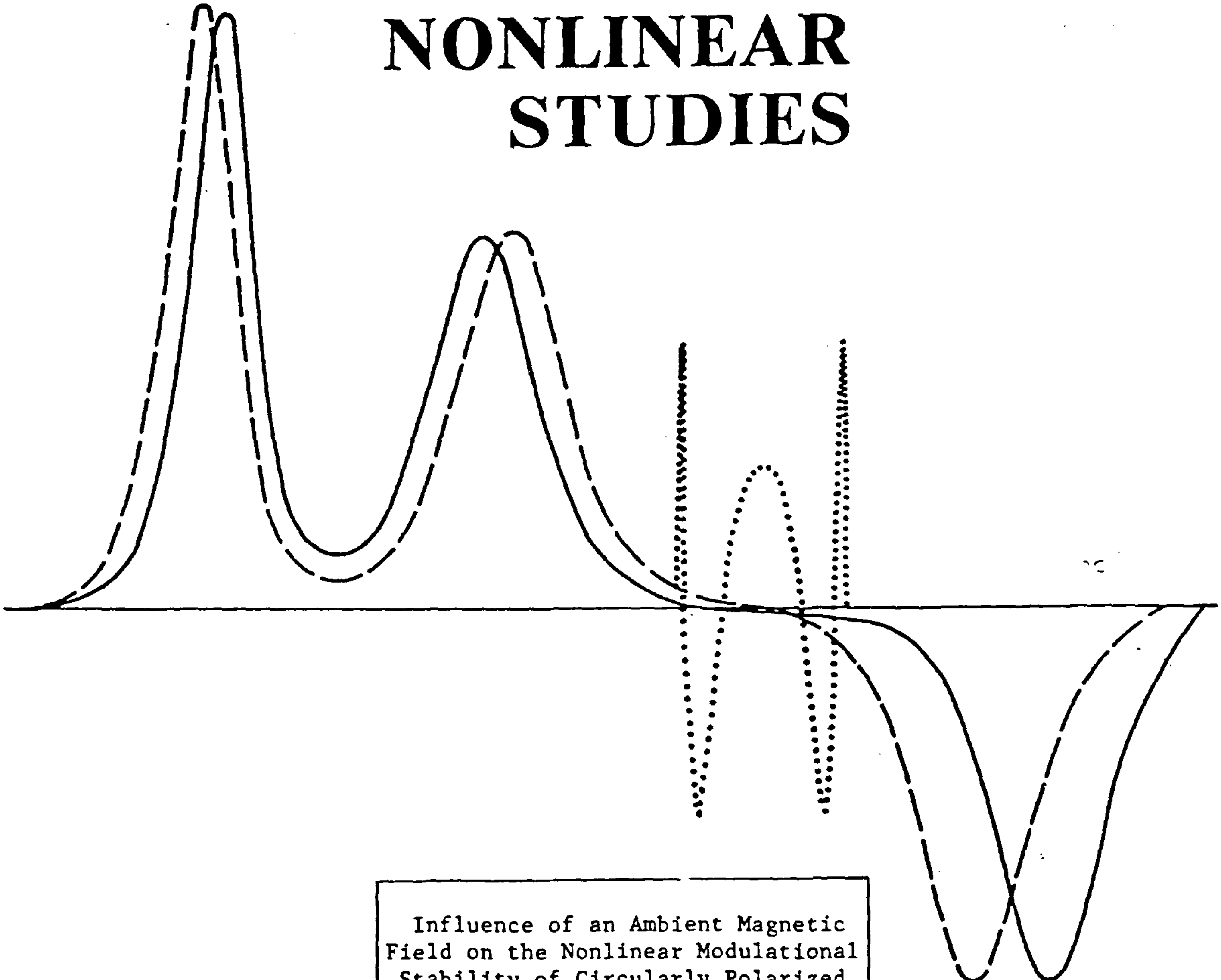

Influence of an Ambient Magnetic Field on the Nonlinear Modulational Stability of Circularly Polarized Electromagnetic Pulses in a Two-Component, Neutral Plasma

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INFLUENCE OF AN AMBIENT MAGNETIC FIELD ON THE NONLINEAR MODULATIONAL STABILITY OF CIRCULARLY POLARIZED ELECTROMAGNETIC PULSES IN A TWO-COMPONENT, NEUTRAL PLASMA

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Abstract

This paper extends our previous results [1] on the nonlinear modulational stability properties of plasma electromagnetic pulses, to include the presence of an ambient magnetic field B_0 parallel to the direction of propagation. As before, the pulse is assumed to be strong enough to accelerate particles to weakly, but not fully, relativistic velocities. The positive component may consist of either positrons or singularly charged ions, and no specific assumptions or approximations are made concerning the mass ratio of the components. (Previous work assumed a positron-electron plasma.) The plasma is assumed fully ionized. The effects of a finite temperature are included for generality. Using singular perturbations, we derive approximate solutions, which describe the evolution of a circularly polarized pulse. The envelope is shown to evolve over long time scales according to the cubic nonlinear Schroedinger (NLS) equation. Relativistic corrections and ponderomotive forces both contribute terms cubic in the amplitude. (In contrast to the case studied in [1], harmonic effects vanish identically here because of circular polarization.) A positron-electron plasma <u>without</u> magnetic fields was shown in our previous paper to be modulationally <u>stable</u>, except in the case of finite temperature where modulational instability is possible near the plasma frequency ω_p. Here, it is shown that, even for a <u>cold</u> plasma, the presence of an ambient magnetic field makes a decisive difference: Modulational instability can arise within a broad range of frequencies and values of B_0, in particular for a pure positron-electron plasma. For given B_0 and polarization, we demonstrate the existence of critical frequencies for the onset of modulational instability. This result has important consequences for observations of pulsar micropulses and possible technological applications.

The possible influence of ambient magnetic fields on the modulational (in)stability of electromagnetic pulses in weakly relativistic plasmas has taken on new importance with the recent finding [1] that a cold positron-electron plasma without ambient magnetic fields is modulationally stable for both linear and circular polarization, contrary to previous findings [2-5]. However, there is strong recent observational evidence [6-9] that pulsar microstructure may result from modulations of a coherent pulse. If nonlinear modulational instability is the cause of micropulses, then some additional element must enter the picture: e.g., finite temperature, ambient magnetic fields, possible presence of ions, or fully relativistic amplitudes. As shown in [1], finite temperature can indeed induce modulational instability for both circular and linear polarization, but the effect is confined to frequencies near the plasma frequency. In this paper, we show that in the presence of an ambient magnetic field, a circularly polarized pulse propagating parallel to the magnetic field in a cold plasma can go modulationally unstable under a range of conditions.

A previous attempt [5] to determine conditions for soliton solutions in a strongly magnetized plasma began from a linear solution which unfortunately contained errors, as pointed out in [1]. These errors were partially corrected in [10]. However, the results apply only to the special case of a single-soliton solution. Correct formulas for the linear theory can be found in [11-12]. As applied in [2-5], the formulas given by Karpman and Krushkal [13] (based on Whitham theory [14]) resulted in expressions which do not satisfy the Maxwell equations and the Lorentz force law, and the nonlinear coefficients of the NLS equation were as a result incorrect.

For these reasons, we have applied the singular perturbation method ("two-timing") used in [1] for the purpose of accurately evaluating the nonlinear coefficnets. (For economy, we refer the reader to the explanations given there.) Since the physical size of a term is explicitly taken into account in assigning orders of smallness, the method contains internal checks which do not allow one to "lose" terms. We have employed the computer algebra system MUMATH to verify that all equations are indeed satisfied at the orders claimed. (The two-timing techniques of [1] were automated by rewriting the usual differentiation routines.) References to equation numbers from [1] in the following will be preceeded by Roman numeral I.

Following [1], our model comprises a singly ionized, positively charged species of mass M and a negatively charged species of mass m, hereafter refered to as the ions and electrons, respectively. (However, since no approximation is made at this stage with respect to m/M, all formulas will be equally valid for an electron-positron plasma upon substituting $M = m$.)

The fluid is characterized as in [1] by the electron density n, the electron velocity $\vec{v}$, the ion density N, the ion velocity $\vec{V}$ the vector potential $\vec{A}$ and the electrostatic potential ϕ. The function n, $\vec{v}$, N, $\vec{V}$, and ϕ as well as the perturbed part of the vector potential $\vec{A}$ are assumed to depend on t and z, but not on x and y. The unperturbed part of $\vec{A}$ describes a uniform magnetic field in the z-direction and thus of course depends linearly on x and y.)

In addition to the plasma frequency, given by

$$\omega_p^2 = 4\pi e^2 n_0 \left(\frac{1}{m} + \frac{1}{M}\right) \tag{1}$$

we define the electron-cyclotron frequency

$$\omega_- = e|B_0|/(mc) \tag{2a}$$

and the ion-cyclotron frequency

$$\omega_+ = e|B_0|/(Mc) \tag{2b}$$

where c is the speed of light. (The two cyclotron frequencies obviously coincide in the case of an electron-positron plasma.)

As in [1], we suppose that the transverse electric field $\vec{E}_\perp$ of the electromagnetic pulse satisfies

$$e|\vec{E}_\perp|/(mc) = O(\epsilon), \tag{3}$$

where ϵ is a dimensionless small parameter and $\epsilon << 1$. As a consequence, the relativistic factors γ and Γ (defined below) differ from unity by $O(\epsilon^2)$.

It is convenient to express the continuity and Euler equations for the electrons in the form

$$\partial_t n + \vec{\nabla} \cdot (n\vec{v}) = 0 \tag{4a}$$

$$d(m\gamma\vec{v} - \frac{e}{c}\vec{A}) - e\vec{\nabla}\phi + m{c_s}^2 \frac{1}{n}\vec{\nabla}n + \frac{e}{c}\Sigma_{\ell=1}^3 v_\ell \vec{\nabla} A_\ell = 0 \tag{4b}$$

where

$$d = \partial_t + \vec{v} \cdot \vec{\nabla} \tag{4c}$$

$$\gamma = (1 - v^2/c^2)^{-1/2} \tag{4d}$$

and c_s is the electron thermal speed. For the ions, the respective equations are

$$\partial_t N + \vec{\nabla} \cdot (N\vec{V}) = 0 \tag{5a}$$

$$D(M\Gamma\vec{V} + \frac{e}{c}\vec{A}) + e\vec{\nabla}\phi + M{C_s}^2 \frac{1}{N}\vec{\nabla}N - \frac{e}{c}\Sigma_{\ell=1}^3 V_\ell \vec{\nabla} A_\ell = 0 \tag{5b}$$

where

$$D = \partial_t + \vec{V} \cdot \vec{\nabla} \tag{5c}$$

$$\Gamma = (1 - V^2/c^2)^{-1/2} \tag{5d}$$

and C_s is the ion thermal speed. Of course, Maxwell's equations

$$\vec{\nabla} \times \vec{B} = \frac{4\pi e}{c}(N\vec{V} - n\vec{v}) + \frac{1}{c}\,\partial_t \vec{E} \tag{6a}$$

$$\vec{\nabla} \cdot \vec{E} = 4\pi e(N - n) \tag{6b}$$

must hold, where we take

$$\vec{B} = \vec{\nabla} \times \vec{A} \tag{6c}$$

$$\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\partial_t \vec{A} \tag{6d}$$

and where we impose the radiation gauge

$$\vec{\nabla} \cdot \vec{A} = 0 \quad . \tag{6e}$$

An exact static solution of the system (4–6) is given by

$$n = N = n_0 \tag{7a}$$

$$\vec{v} = 0 \tag{7b}$$

$$\vec{V} = 0 \tag{7c}$$

$$\phi = 0 \tag{7d}$$

$$\vec{A} = \vec{A_0} \equiv (-yB_0/2, xB_0/2, 0) \tag{7e}$$

which contains a constant magnetic field in the z direction. Quantities with subscript 0 are strictly constant.

Following [1], we assume asymptotic expansions of the form

$$n = n_0 + \epsilon n_1 + \epsilon^2 n_2 + \epsilon^3 n_3 + \dots \tag{8a}$$

$$\vec{A} = \vec{A_0} + \epsilon \vec{A_1} + \epsilon^2 \vec{A_2} + \epsilon^3 \vec{A_3} + \dots \tag{8b}$$

$$N = n_0 + \epsilon N_1 + \epsilon^2 N_2 + \epsilon^3 N_3 + \dots \tag{8c}$$

$$\vec{v} = \epsilon \vec{v_1} + \epsilon^2 \vec{v_2} + \epsilon^3 \vec{v_3} + \dots \tag{8d}$$

$$\vec{V} = \epsilon \vec{V_1} + \epsilon^2 \vec{V_2} + \epsilon^3 \vec{V_3} + \dots \tag{8e}$$

$$\phi = \epsilon \phi_1 + \epsilon^2 \phi_2 + \epsilon^3 \phi_3 + \dots \tag{8f}$$

where A_0 is given by (7e).

It is instructive to review briefly properties of the solutions of the linearized system obtained by simply truncating Eqs. (4–6) at $O(\epsilon)$. (The linearized theory of circularly-polarized electromagnetic-wave propagation parallel to an ambient magnetic field can be found in [12].) A typical Fourier component of the vector potential perturbation behaves in this linearized theory like $\vec{a}(\omega)e^{i(kz-\omega t)}$, where

$$\vec{a} = a \cdot (\hat{x} + i\hat{y}). \tag{9}$$

The dispersion relation for right-hand circular polarization is

$$\omega^2 = 4\pi e^2 n_0/(m\nu) + 4\pi e^2 n_0/(M\mu) + k^2 c^2 \tag{10}$$

where

$$\nu \equiv 1 - eB_0/(mc\omega) = 1 - \omega_-/\omega \tag{11a}$$
$$\mu \equiv 1 + eB_0/(Mc\omega) = 1 + \omega_+/\omega \tag{11b}$$

The dispersion relation for left-hand circular polarization can be obtained from (11) by substituting for ν and μ expressions in which B_0 is replaced by $-B_0$.

If the masses are unequal, then four electromagnetic branches can be distinguished. If the masses are equal (electron-positron plasma), then two branches are present. The requirements $k^2 > 0$ and $\omega^2 > 0$ will in general place restrictions on the allowable frequencies of a propagating electromagnetic wave. Unlike the unmagnetized case, propagation can occur below the plasma frequency. In the linear theory it is of course also possible to consider the evolution of initial data corresponding to "linear" polarization; evidently Faraday rotation of the polarization will occur.

In addition to the electromagnetic modes, there are also longitudinal Langmuir oscillations at the plasma frequency. In the linearized theory, these are of course decoupled from electromagnetic waves. However, in the modulated nonlinear theory we shall see that the ponderomotive force induces coupling between transverse and longitudinal modes.

Following the procedure of [1], we now assume that the first-order perturbation of the vector potential and electron density take the form

$$\vec{A}_1 = \vec{a}e^{i(kz-\omega t)} + c.c. \tag{12}$$
$$\vec{a} = a \cdot (\hat{x} + i\hat{y}) \tag{13}$$
$$n_1 = 0, \tag{14}$$

where "c.c." indicates the complex conjugate, and where the wave number k and the frequency ω correspond to nonsingular solutions of the dispersion relation (10). In particular we assume that ω is quite distinct from the electron-cyclotron frequency and also the ion-cyclotron frequency. (In fact, we exclude a small range of $O(\epsilon)$ around these frequencies). The function a is not strictly constant, but depends on the slow variables

$$Z \equiv \epsilon z, \quad T \equiv \epsilon t, \quad \tau \equiv \epsilon^2 t \tag{15}$$

This choice of the form of the first-order perturbation places restrictions on the class of initial conditions described by our treatment: The initial conditions correspond to a right-hand circularly polarized pulse propagating in the positive z direction. (Results for the opposite polarization can be recovered by replacing B_0 with $-B_0$ at the end.)

As in [1], in first order, the longitudinal components are identically zero or can be chosen to vanish without loss of generality. The transverse components can be treated as a coupled linear homogeneous system of the form

$$L(\psi_1) = 0 \quad . \tag{16}$$

The solution can be expressed in the form

$$\phi_1 = 0 = n_1 = N_1 = A_{1z} = v_{1z} = V_{1z} \tag{17a}$$
$$\vec{v}_1 = e^{i(kz-\omega t)} e\vec{a}/(mc\nu) + c.c. \tag{17b}$$
$$\vec{V}_1 = -e^{i(kz-\omega t)} e\vec{a}/(Mc\mu) + c.c. \tag{17c}$$

The notation $\vec{v}_1$ means the the $O(\epsilon)$ part of $\vec{v}$, and so forth. The dispersion relation (10) of the (truncated) linearized theory must also hold. The first-order part of (6b) guarantees quasineutrality at first order. The z-component of (6a) is then satisfied at first order automatically.

We now refer the reader to Eqs. (I.15)-(I.18) and discussion. As before, the second-order parts of the transverse equations – together with the condition for no secular terms – restrict the slow dependence of a. One finds explicitly that the amplitude is transported according to

$$(\partial_T + v_g \partial_Z)\, a = 0 \tag{18a}$$

at the group velocity

$$v_g = d\omega/dk \tag{18b}$$

where

$$v_g = \frac{kc^2}{\omega}\left(1 + \frac{2\pi e^3 n_0 B_0}{\omega^3 c}\left[1/(m^2\nu^2) - 1/(M^2\mu^2)\right]\right)^{-1} \tag{18c}$$

where ν and μ were defined in Eq. (11). Note that the product of group and phase velocity is not c^2. We now examine the next leading order in the remaining equations. These consist of third order in the longitudinal parts of the Euler equations (4b,5b), second order in the continuity equations (4a,5a), second order in Poisson's equation (6b), and second order in the transverse Euler equations (4b,5b). (The longitudinal part of (6a) is then satisfied automatically.)

We recall that in [1], the second-order solution contained three types of terms classified according to their phase dependence as "DC," "fundamental," and "harmonic." All three types of terms contributed effects of comparable magnitude, in the general case. However, harmonic terms vanished in the special case of circular polarization. The same is true here, even though a magnetic field is present. On the other hand, ponderomotive forces arise in the longitudinal equations and induce DC corrections as before.

The second-order equations form a linear inhomogeneous system, and thus given any particular second-order solution one can find another particular solution (corresponding to different initial data, of course) by the addition of an arbitrary homogeneous solutions of the form (16).

$$\vec{A}_2 = 0, \tag{19}$$

we obtain for the DC part

$$n_2 = aa^* n_0 \sigma \tag{20a}$$

$$N_2 = n_2 \tag{20b}$$
$$v_{2z} = v_g a a^* \sigma \tag{20c}$$
$$V_{2z} = v_g a a^* \sigma \tag{20d}$$
$$\phi_2 = a a^* \Sigma \tag{20e}$$

where

$$\sigma = \frac{e^2(\omega^2 + c^2k^2 - 2kv_g\omega)}{c^2\omega_p^2 m M {v_g}^2 F} \tag{21}$$

$$\Sigma = \left(\frac{e}{(M+m)c^2{v_g}^2F}\right)\left(\frac{M}{m\nu}({v_g}^2 - {C_s}^2)\left[1 + \frac{kv_g\omega_c}{\omega^2\nu}\right] - \frac{m}{M\mu}({v_g}^2 - {c_s}^2)\left[1 - \frac{kv_g\Omega_c}{\omega^2\mu}\right]\right) \tag{22}$$

$$F = 1 - \frac{m{c_s}^2 + M{C_s}^2}{{v_g}^2(M+m)} \tag{23}$$

Note that F contains the corrections due to finite temperature. Also note that longitudinal perturbations are forced by the modulations in a. As in [1], charge quasi-neutrality at second order still holds. However, in contrast to [1] the second-order transverse velocity corrections are nonvanishing. These are

$$\vec{v}_{2\perp} = -ie^{i(kz-\omega t)}e^2 B_0 \partial_T \vec{a}/(m^2\omega^2\nu^2) + c.c. \tag{24a}$$
$$\vec{V}_{2\perp} = -ie^{i(kz-\omega t)}e^2 B_0 \partial_T \vec{a}/(M^2\omega^2\mu^2) + c.c. \tag{24b}$$

We now proceed to third order. Choosing the initial conditions as above to exclude <u>homogeneous</u> Langmuir oscillations and again absorbing $\vec{A}_3$ into the first order, we solve the transverse parts of the Euler equations (4b,5b) to obtain the particular solution

$$\vec{v}_{3\perp} = e^{i(kz-\omega t)}(\hat{x} + i\hat{y})s_3 + c.c. \tag{25a}$$
$$\vec{V}_{3\perp} = e^{i(kz-\omega t)}(\hat{x} + i\hat{y})S_3 + c.c. \tag{25b}$$

where s_3 is given by

$$\begin{aligned} s_3 = &- ie\omega_-\partial_\tau a/(mc\omega^2\nu^2) - e\omega_-\partial_T^2 a/(mc\omega^3\nu^3) \\ &+ ekv_g\omega_- a^2a^*\sigma/(mc\omega^2\nu^2) - 2e^3a^2a^*/(m^3\nu^4) \end{aligned} \tag{26}$$

An expression for S_3 can be obtained from s_3 by replacing m with M, ν with μ, ω_- with ω_+, and e with $-e$. The last term in (26) is the relativistic correction and except for the corresponding term in S_3, it is the only relativistic correction. The other nonlinear term in (26) is a ponderomotive-like term arising from the second-order shift in the longitudinal velocity, v_{2z}.

The longitudinal equations at the next leading order give information about the third-order density and potential corrections. However, we already have all the information necessary to evaluate the secular condition implied by the transverse components of the Maxwell equation (6a). The linear terms are consistent with what would have been obtained by the Karpman-Krushkal [13] approach. In particular, the coefficient of $\partial_\tau a$ can be shown to be equal to $2kc^2/v_g$, The dispersion terms (second derivatives in T and/or Z) can be collected together by means of (18a) into a coefficient times $\partial_Z^2 a$. This coefficient turns out to be $kc^2\omega_{kk}/v_g$, where ω_{kk} is called the dispersion and is the second derivative of ω with respect to k. To be sure, considerable algebra is required in order to obtain the final result. However, there are some "miraculous cancellations" and things do simplify toward the end. After division by some common factors, we finally arrive at an equation of the form

$$2i\omega\partial_\tau a + \omega\omega_{kk}\partial_Z^2 a + a^2 a^*(C_P + 2C_R) = 0 \tag{27}$$

where the nonlinear coefficient C_R contains the effects of the relativistic corrections, while the nonlinear coefficient C_P contains the ponderomotive effects. The latter arises from two sources. First there is the shift in the second-order longintudinal velocity mentioned above. This term did not occur in the zero-magnetic-field case [1]. The other source is the second-order shift in the density which did occur in the zero-magnetic-field case [1]. Finally, note that no terms of the type refered to in [1] as "harmonic" occur here (such terms would be expected for generic polarization).

The sum of the two ponderomotive terms is given simply by

$$C_P = -\left(\frac{e^2}{c^4 Mm}\right)\frac{\omega}{k\omega_p^2 v_g F}(2k\omega v_g - \omega^2 - c^2k^2)^2 \tag{28a}$$

The relativistic coefficient is given explicitly by

$$C_R = \left(\frac{e^2}{c^4 Mm}\right)\frac{\omega v_g M^2 m^2 \omega_p^2}{2c^2 k(M+m)}\left[1/(m^3\nu^4) + 1/(M^3\mu^4)\right] \tag{28b}$$

We note that the coefficients C_P and C_R reduce to the corresponding formulas (I.25b-c) if B_0 vanishes.

The condition for modulational instability is

$$\omega\omega_{kk}(C_P + 2C_R) < 0 \tag{29}$$

It is remarkable that C_R can in fact be reduced to a functional of the linear dispersion relation alone. One can show that (28b) is equivalent to

$$C_R = \left(\frac{e^2}{c^4 Mm}\right)\frac{\omega^5 v_g}{2c^2 k\omega_-\omega_+}\left[\frac{1}{6}\omega\partial_\omega{}^3(c^2k^2) - 1 + \frac{1}{2}\partial_\omega{}^2(c^2k^2)\right] \tag{30}$$

where k is regarded here as an implicit function to be obtained by solving the dispersion relation for k in terms of ω. The practical advantage of (30) (together with (28a)) is that it reduces the problem of evaluating the NLS coefficents to formulas involving only the linear dispersion relation. Given $c^2k^2(\omega)$, one may now evaluate all of the NLS coefficients. Thus the problem of calculating the nonlinear coefficents is simply a problem in evaluating v_g and $\partial_\omega^3(c^2k^2)$. These expressions could be useful in obtaining clues to the corresponding result for arbitrary polarization.

We examine the electron-positron case in detail. In this case, the dispersion relation (10) reduces to

$$c^2k^2 = \omega^2\left[1 - \frac{\omega_p^2}{\omega^2 - \Omega^2}\right] \tag{31}$$

where Ω is now the common cyclotron frequency. The group velocity is found to be

$$v_g = \left(\frac{c^2k}{\omega}\right)\frac{1}{1 + \frac{\omega_p^2\Omega^2}{(\omega^2-\Omega^2)^2}} \tag{32}$$

and the nonlinear coefficients are

$$C_R = \left(\frac{e^2}{c^4m^2}\right)\frac{\omega_p^2\omega^5 v_g}{2kc^2(\omega^2-\Omega^2)^2}\left[1 + \frac{8\omega^2\Omega^2}{(\omega^2-\Omega^2)^2}\right] \tag{33a}$$

$$C_P = -\left(\frac{e^2}{c^4m^2}\right)\frac{\omega_p^2\omega^7 v_g}{k^3c^4(\omega^2-\Omega^2)^2F}\left[1 + \frac{2\Omega^2}{\omega^2-\Omega^2} - \frac{\omega_p^2\Omega^2}{(\omega^2-\Omega^2)^2}\right]^2 \tag{33b}$$

while the dispersion is

$$\omega_{kk} = \frac{v_g^3\omega^2\omega_p^2}{c^4k^3(\omega^2-\Omega^2)}\left[1 + \frac{5\Omega^2}{(\omega^2-\Omega^2)} + \frac{\Omega^2(4\Omega^2-3\omega_p^2)}{(\omega^2-\Omega^2)^2} - \frac{3\omega_p^2\Omega^4}{(\omega^2-\Omega^2)^3}\right]. \tag{34}$$

In the limit of large Ω^2/ω_p^2, which is certainly a good approximation for the pulsar environment, one obtains from the above, for $k/\omega > 0$,

$$v_g = c(1 - \frac{\omega_p^2}{2\Omega^2} + \cdots) \tag{35a}$$

$$\omega_{kk} = -\frac{3\omega\omega_p^2c^2}{\Omega^4} + \cdots \tag{35b}$$

$$C_R = \left(\frac{e^2}{c^4m^2}\right)\frac{{\omega_p}^2\omega^5 v_g}{2kc^2\Omega^4}\left(1 + \frac{10\omega^2}{\Omega^2} + \cdots\right) \tag{35c}$$

$$C_P = -\left(\frac{e^2}{c^4m^2}\right)\frac{{\omega_p}^2\omega^5 v_g}{kc^2\Omega^4F}\left(1 + \frac{6\omega^2 + {\omega_p}^2}{\Omega^2}\right) + \cdots) \tag{35c}$$

$$F = 1 - \frac{c_s^2 + C_s^2}{2c^2} + \cdots \tag{35e}$$

Note that (35b) gives $\omega_{kk\omega} < 0$. In the limit of $B_0 \to 0$, we have the opposite result of $\omega_{kk\omega} > 0$. Recall that in I, it was positive nonlinear coefficients which led to modulational instability; here it will be negative nonlinear coefficients which will give rise to modulational instability.

As long as the sound speeds are much smaller than the group velocity of the electromagnetic wave, (which is usually the case for nonrelativistic gas temperatures) F is close to unity. (However, the A and FMS branches [12] of the dispersion relation have vanishing group velocity near the cyclotron frequencies, and therefore for finite temperature sufficiently near these frequencies F may become significantly different from unity.) Using Eqs. (35), we find

$$C_P + 2C_R = \left(\frac{e^2}{c^2 m^2}\right) \frac{{\omega_p}^2 \omega^4}{2\Omega^4} \left[\frac{4\omega^2 - \omega_p^2}{\Omega^2} - \frac{c_s^2 + C_s^2}{2c^2} + \cdots\right] \tag{36}$$

Now we can have modulational instability, see (29), only if

$$\omega^2 < \frac{1}{4}\omega_p^2 + \Omega^2 \frac{c_s^2 + C_s^2}{2c^2} \tag{37}$$

For large Ω^2, we see that there is a very wide frequency range for even moderate temperatures. And even for a cold plasma, modulational instability occurs also if $\omega^2 < \frac{1}{4}{\omega_p}^2$.

According to pulsar models [15,16], the pulsar magnetosphere is composed of secondary electrons and positrons, and strong magnetic fields (10^{12} Gauss) are present, corresponding to an electron plasma frequency of the order 10^{19} Hz, as compared to a plasma frequency of a few Megahertz. If we apply the statistical model of Rickett [17], as described in [2], a modulationally unstable pulse could be responsible for the micropulse structure observed in [18].

Unfortunately, one cannot yet make any definitive predictions for the problem of micropulses. First, one would expect that the propagation angle will in general not be parallel to the magnetic field. Second, the polarization may not necessarily be circular. Third, the pulses may well be fully relativistic. Still, based on the present calculation, one would expect that a range of frequencies for modulational instability should exist even when the propagation angle is not parallel to the magnetic field, and even when the polarization is elliptic. (Perhaps the observed mixed polarizations result from averaging due for example to insufficiently fine time resolution.) Consideration of fully relativistic effects will have to await the results of further work.

Our paper predicts a definite frequency range for instability; for the case of a positron-electron plasma, the frequency range is given directly in Eq. (37). For the more general case, the frequency range can easily be obtained from Eqs. (28) and (29), together with the dispersion relation (10).

A further result of our calculation is a prescription for obtaining the nonlinear coefficients directly from the linear dispersion relation, as in Eqs. (30) and (28a) above. Note that these two equations, although obtained for a two component plasma, have an obvious generalization for a plasma containing any number of components. Thus, given an N-component neutral plasma and a linear dispersion relation, one can directly obtain the coefficients of nonlinearity in the case of circular polarization. This is in the spirit of the Karpman-Krushkal [13] method, which Refs. [2-5] and [10] attempted to apply.

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